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THE undersigned, acting as a committee of the Graduate School. have read the accompanying thesis submitted by Vera Linn Wright for the degree of Master of Arts They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts.

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191

A STUDY OF THE RIGHT HELICOID.

A THESIS

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OF THE

GRADUATE SCHOOL OF THE UNIVERSITY OF MINNESOTA

BY

VERA L. WRIGHT

FOR THE

DEGREE OF MASTER OF ARTS.

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- BIBLIOGRAPHY -

Eisenhart, L. P. "A Treatise on the Differential Geometry of Curves and Surfaces." Ginn & Co.

Joachimsthal F. "Anwendung der Differential - und Integralrechnung." Leipzig. Druck und Verlag von B. G. Teubner, 1890.

Kiepert und Stegemann "Grundriss der Differential - und Integral-Rechnung." Hanover. Helwingsche Verlagsbuchhandlung, 1901.

Kommerell und Kommerell. "Allgemeine Theorie der Raumkurven und Flächen." G. J. Göschensche Verlagshundlung, 1903.

Scheffers, G. "Anwendung der Differential - und Integral - Rechnung auf Geometrie." Leipzig. Verlag von Veit & Co. 1913.

A STUDY OF THE RIGHT HELICOID.

I. The Generation of the Helicoid.

The helicoid is a surface generated by a curve, plane or twisted, which is rotated about a fixed line as an axis and at the same time is translated in the direction of the axis with a velocity that bears a constant ratio to the rate at which the curve is being rotated. The right helicoid, the surface under discussion in this paper, is that particular type of helicoid in which the generator is a straight line perpendicular to the axis of rotation.[1]

II. The Derivation of the Equations of the Helicoid.

The derivation of the equations of the helicoid depends directly on the equations of a circular helix which is defined in the following manner.[2] As in the figure(1),

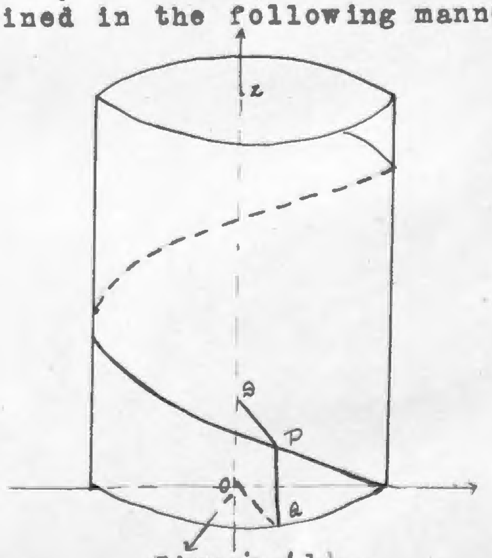


Figure (1).

[1] Note. In the remainder of the paper the term helicoid is used only in the sense of right helicoid.

[2] For the method used see Eisenhart pages 1 and 2.

let a line segment PD of constant length a and perpendicular to a line OZ at D be revolved about OZ as an axis. If PD is translated in the direction of OZ with a velocity bearing a constant ratio to its rate of rotation about OZ, the locus of P is called a circular helix. To express the equations of this locus in the parameter form, let OZ be taken as the z axis and the initial position of PD as the x axis. Also, let the angle between the initial position of PD and any of its subsequent positions be denoted by v. Then the equations of the circular helix may be written in the form,

(1) $x = a\cos v, \quad y = a\sin v, \quad z = kv$

where k is a constant determined by the velocity of the translation of D with respect to the rate of the rotation of PD. The first two equations express the condition that P lies on a cylinder and the third that P rises uniformly on the cylinder.

If in the equations (1), the constant length a is replaced by a variable u, the equations are of the form

(2) $x = u\cos v, \quad y = u\sin v, \quad z = kv.$

That these equations (2) represent the helicoid is evident; for, since the point P has been replaced by a line of length u, the equations are those of a surface generated by a line perpendicular to an axis and simultaneously rotated about and translated in the direction of the axis. The constant k has the same significance in the equations (2) as in the equations (1), while by analogy v in the equations (2) denotes the angle which a plane thru the generating line and the z axis makes with the xz plane. The parameter lines are then, the straight lines which represent each position of the generating line and the helices which any point of the generator traces. This may be seen analytically by letting v equal a constant, in which case the equations

(3) $x = au \quad , \quad y = bu \quad , \quad z = k'$,

a, b, and k' denoting constants, are those straight lines perpendicular to the z axis, i. e. are the generators of the surface; similarly by letting u equal a constant the equations of a circular helix,

(4) $x = c\cos v \quad , \quad y = c\sin v \quad , \quad z = kv$,

are obtained for every value of the constant c. These parameter lines are represented in the figure (2).

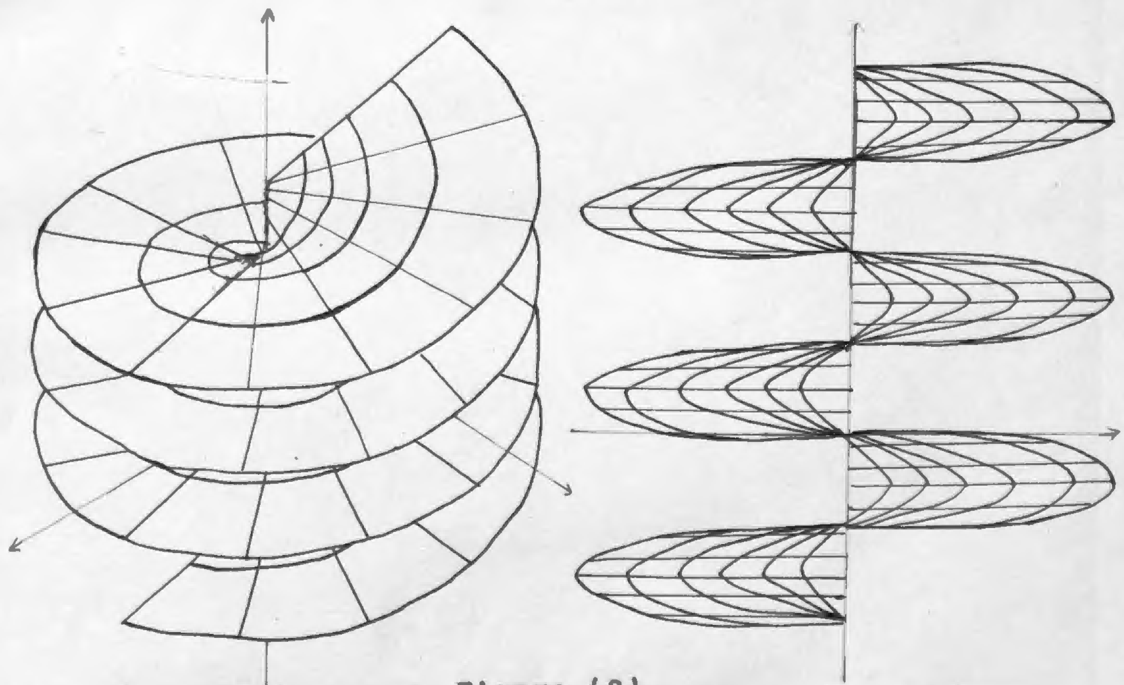


Figure (2).
Same surface from two points of view.

III. The Helicoid - A Ruled Surface.

Since as seen in part II there is a system of straight lines on the helicoid, it may be thought of as a ruled surface. It is easily shown that the principal normals to a helix are perpendicular to and intersect the axis of the helix. Thus the helicoid is a skew ruled surface whose generators are the principal normals to the directrix curve which is a helix.

IV. The Values of the Fundamental Quantities of the Helicoid.

The computation of the first and second set of fundamental quantities, the element of arc and the mean and total curvatures of the helicoid[1] as given by the equations (2) leads to the following results:

$$E = 1 \quad , \quad F = 0 \quad , \quad G = u^2 + k^2$$

$$L = 0 \quad , \quad M = \frac{-k}{\sqrt{u^2+k^2}} \quad , \quad N = 0$$

$$ds^2 = du^2 + (u^2+k^2)\, dv^2$$

$$K = \frac{1}{\rho_1 \rho_2} = \frac{-k^2}{(u^2+k^2)^2} \quad , \quad H = \frac{1}{\rho_1} + \frac{1}{\rho_2} = 0.$$

V. The Asymtotic Lines on the Helicoid.

Since L and N are zero the asymtotic lines on the helicoid are the parameter lines themselves[2] the discussion of which was given in part II.

(1) For the definitional values of these quantities see Scheffers Part II: Anhang Tafel XI and XII.
(2) Eisenhart page 129.

VI. The Curvature at Any Point of the Helicoid.

From the value of H as given in part IV, it follows that each point of the surface the curvature in the two principal normal sections are equal in absolute value but of unlike sign and the helicoid has therefore hyperbolic curvature at each point and is furthermore a minimal surface.

VII. The Equations of the Lines of Curvature on the Helicoid.

The equations by means of which the lines of curvature are given may be obtained by substituting in the formula[1]

$$\begin{vmatrix} -dv^2 & E & L \\ -du\,dv & F & M \\ du^2 & G & N \end{vmatrix} = 0$$

the values for E, F. G. L. M and N as given in part IV and are of the form

(5) $$dv = \pm \frac{du}{\sqrt{u^2+k^2}}$$

The integration of this gives the equations

(6) $$e^{\pm v} = \frac{u+\sqrt{u^2+k^2}}{a}$$

wherein a is the constant of integration. The two sys-

[1]Scheffers Part II Anhang Tafel XII.

tems of lines of curvature on the helicoid are obtained from the equations (6) and (2).

VII. The Conformal Representation of the Helicoid on a Plane.

The geometrical interpretation of the lines of curvature will be made thru the aid of the conformal representation of the helicoid on a plane. A surface is defined as being represented conformally upon another if between the two surfaces there is a one to one correspondence of points, and if angles between corresponding lines on the surfaces are equal, or in other words if corresponding portions of the surfaces are similar.[1] A set of necessary and sufficient conditions for the conformal representation of one surface upon another are the following:[2]

$$(7) \qquad \frac{E}{E'} = \frac{F}{F'} = \frac{G}{G'} = t^2$$

where the primed quantities refer to the surface on which the first is conformally represented and where the factor of proportionality t^2 is a function of the parameters.

Isometric curves on a surface are parameter curves which are perpendicular and have equal elements of arc or geometrically speaking isometric curves are such that they divide the surface into infinitesimal squares.[3] A set of conditions for isometric parameters on a surface[4] is

$$E = G, \qquad F = 0.$$

If this type of parameter is used on a surface, the sur-

(1) See Eisenhart Article 42.
(2) See Eisenhart Article 42.
(3) See Eisenhart Article 40.
(4) See Scheffers Part II, Satz 25.

face may then be represented conformally on a plane, the parameter lines on the surface may be chosen as corresponding to the polar coordinates of the plane.[1]

Thus in order that the helicoid may be so represented conformally upon the plane, it will be sufficient to express it in isometric parameters. The element of arc of the helicoid given in part IV may be expressed in the following form

$$ds^2 = (u^2+k^2)\left(\frac{du^2}{u^2+k^2} + dv^2\right).$$

If new parameters U and V are introduced by means of the relations

$$U = \int \frac{du}{\sqrt{u^2+k^2}} = \log\left(u+\sqrt{u^2+k^2}\right), \quad V = v.$$

which also may be written as

$$e^U = u+\sqrt{u^2+k^2}, \quad V = v$$

then the element of arc appears in the form

$$(8) \qquad ds^2 = \frac{(e^U - e^{-U})^2}{4}\left(dU^2+dV^2\right)$$

from which it is at once seen that the U, V parameter lines form an isometric system. The equations of the helicoid when u and v are replaced by U and V assume the form

$$x = \frac{1}{2}(e^U + e^{-U})\cos V, \quad y = \frac{1}{2}(e^U + e^{-U})\sin V, \quad z = kV,$$

(1)Scheffers Part I, S. 28

and the equations by means of which the lines of curvature are obtained assume from the equations (6) the form

(10) $$U = \pm c V$$

where c equals log a.

In the course of the following section it will be well to have clearly in mind the correspondence between the curves on the helicoid and the polar coördinates. The U, V parameter curves are again the helices and straight line generators discussed in part II, the only difference being that the helices are differently ordered. Also in order that the r θ parameter curves of the coördinate plane may form an isometric system it is necessary to introduce the new parameters[1]

$$R = \log r, \qquad \Theta = \theta$$

or in other words to rearrange the concentric circles. Hence to the straight line generators ($V = v =$ constant) and the helices ($U = \log (u - \sqrt{u^2 - k^2})$) there correspond in the plane the radio vectors ($\Theta = \theta =$ constant) and the concentric circles ($R = \log r =$ constant).

VIII. The Geometrical Interpretation of the Lines of Curvature on the Helicoid.

The graph of the equations (10) in a plane where U and V are used as polar coordinates is that of a double infinity of spirals similar to those of the figure 3 which, starting out from the origin in opposite directions under the same angles, form the diagonals of the net of coordinate curves.

Since the helicoid is represented conformally on the plane in the manner just described in part VII, the lines of curvature on the former are the curves which as shown in the figure (4) form the diagonals of the net of U V parameter curves.

(1) Scheffers Part I S. 128.

(8-a). —

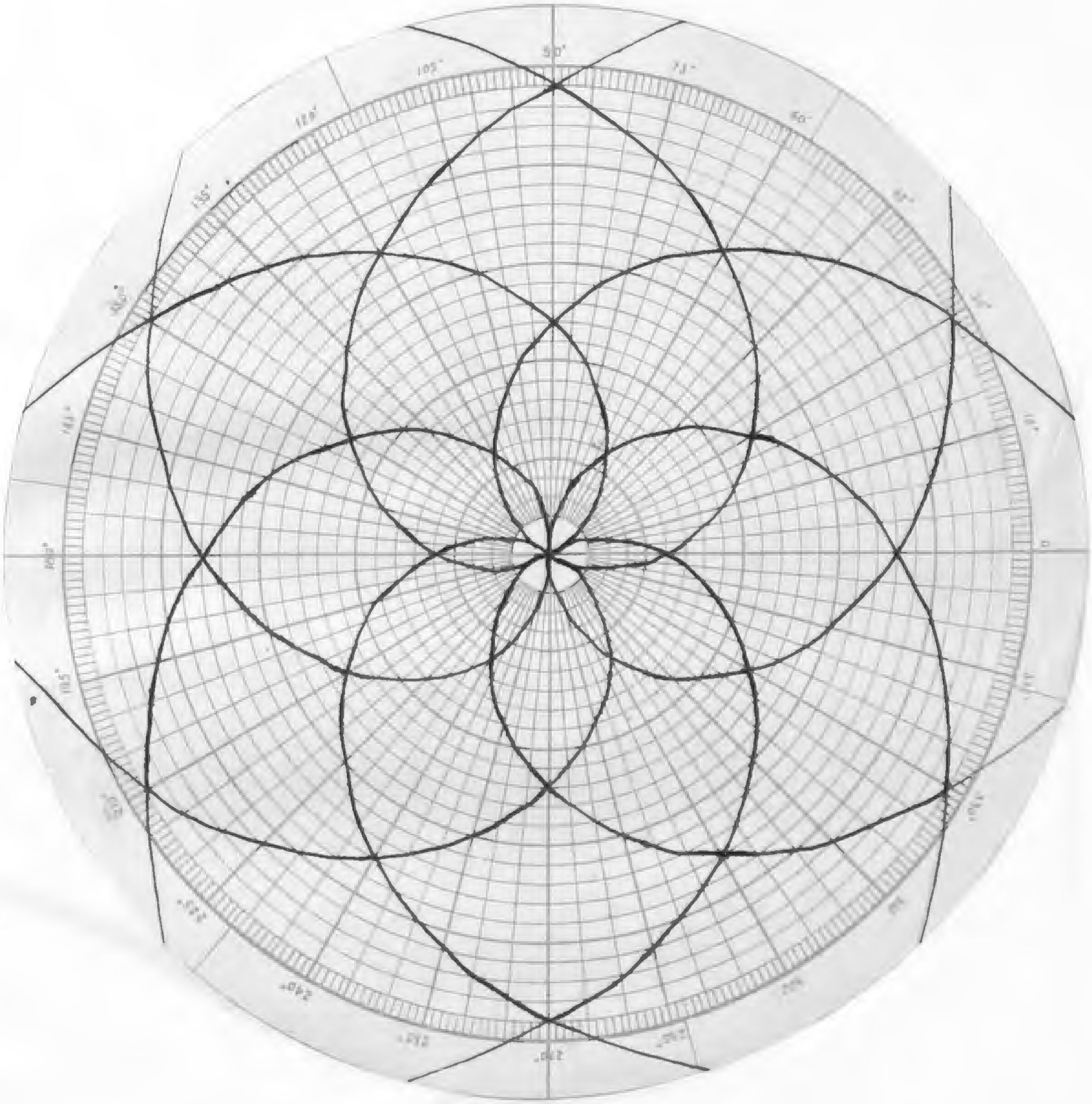


Figure - 9 -

(8b)

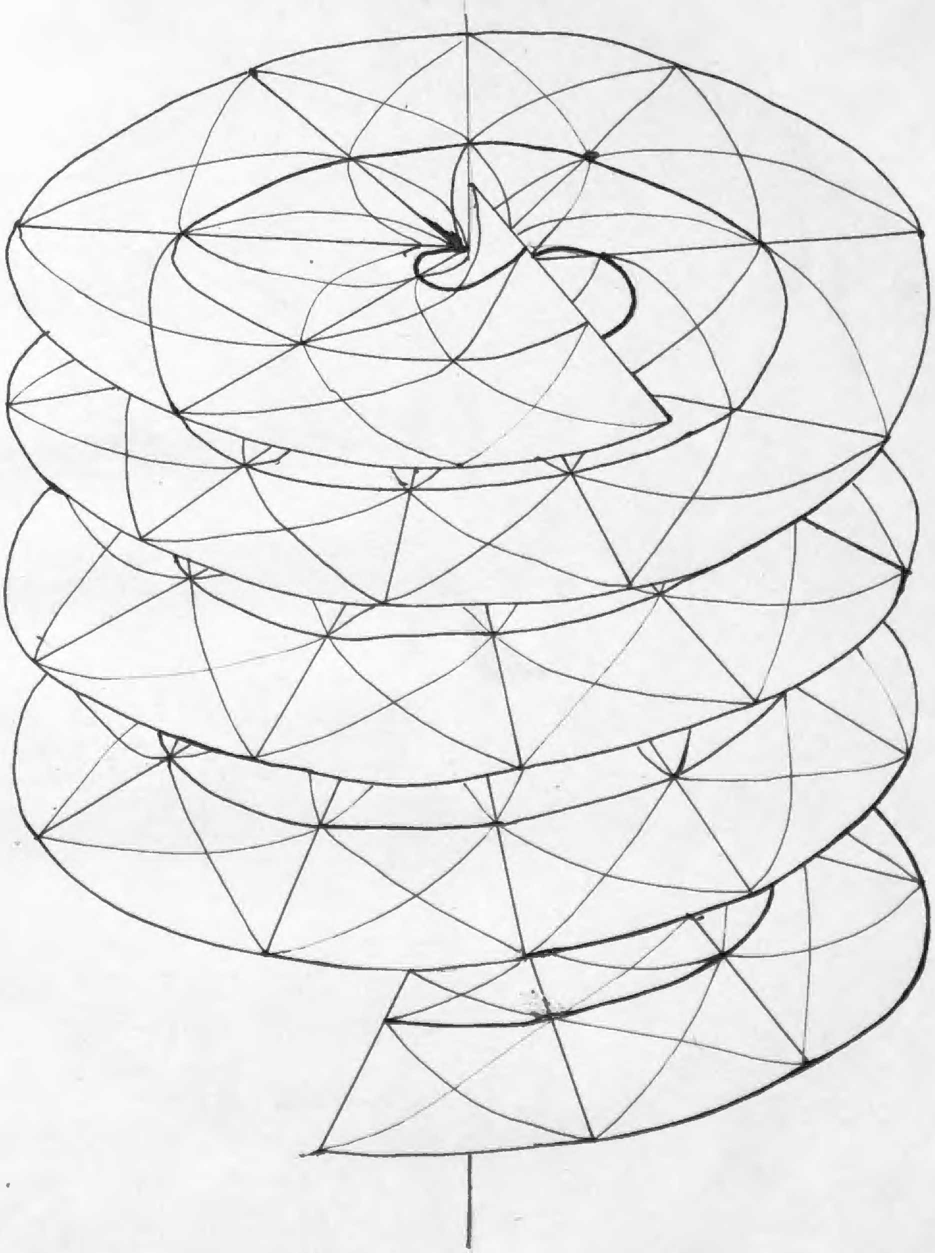


Figure (4).

IX. The Applicability of the Catenoid to the Helicoid.

Since the element of arc of the helicoid, as seen by the equation (7), may be expressed in the form

$$ds^2 = m^2 (dU^2 + dV^2)$$

where m^2 is a function of U alone, the helicoid is applicable to some surface of revolution.[1] The term applicable has the signaficance that the corresponding portions of the two surfaces in the neighborhood of every point instead of being similar as in conformal representation are equal and can therefore be bent so as to coincide. In other words there is a one to one correspondence such that corresponding angles and elements of arc are equal and hence the factor of proportionality of the equation (7) is unity.[2] The determination of the geodesics on the helicoid can be made to advantage by the study of the geodesics on that surface of revolution which is applicable to the helicoid. To find the type of surface of revolution applicable to the helicoid i. e. to determine the form of the function in the equation of any surface of revolution,

$$x = \bar{u}\cos\bar{v}, \quad y = \bar{u}\sin\bar{v}, \quad z = \varphi(\bar{u}) \tag{11}$$

in order that the elements of arc on each surface shall be equal. The following method[3] may be used.

If $\bar{u}$ is equated to m and the resulting equation is solved for U as a function of $\bar{u}$, the value

$$U = \log \frac{\bar{u} + \sqrt{\bar{u}^2 - f^2}}{f}$$

is obtained. The expression for $\varphi(\bar{u})$ may now be derived from the relation

(1) Eisenhart Article 46.
(2) Eisenhart Article 43.
(3) Eisenhart Article 46.

$$\log \frac{\bar{u} + \sqrt{\bar{u}^2 - b^2}}{b_0} = \int \frac{1}{\bar{u}} \sqrt{1 + \varphi'(\bar{u})^2}\, d\bar{u}$$

and has the form

$$\varphi(\bar{u}) = b \log \left(\bar{u} + \sqrt{\bar{u}^2 - b^2} \right).$$

Hence the surface of revolution applicable to the helicoid is given by the equations

(12) $x = \bar{u} \cos \bar{v}, \quad y = \bar{u} \sin \bar{v}, \quad z = b \log \left(\bar{u} + \sqrt{\bar{u}^2 - b^2} \right),$

which are recognized as representing a catenoid of revolution formed by revolving about the z axis the catenary

$$x = \frac{b}{2} \left(e^{\frac{z}{b}} + e^{-\frac{z}{b}} \right).$$

The element of arc of the catenoid assumes the form

$$ds^2 = \frac{\bar{u}^2}{\bar{u}^2 - b^2} \left(d\bar{u}^2 + \bar{u}^2\, d\bar{v}^2 \right).$$

It is at once evident that if the constants k and b are chosen equal and the parameters of the two surfaces are related by the equations

(13) $\bar{u}^2 = u^2 + k^2, \quad \bar{v} = v$

then the elements of arc of the two surfaces are equal.

The next problem is to visualize by means of the relations (13) the manner in which a portion of the catenoid has to be bent in order to make it coincide with a corresponding portion of the helicoid. The length of arc of any of the meridian sections of the catenoid measured from the points on the minimal circle has the value[(1)]

$$s = \sqrt{\bar{u}^2 - k^2},$$

By means of the equations (13) u, the distance from the z axis of the helicoid along any of its generating lines may be expressed in the form

$$u = \sqrt{\bar{u}^2 - k^2}$$

and it is seen that the u of the helicoid is equal to the arc s described above. Hence to a point at a distance s measured along a catenary ($\bar{v}$=c) from the minimal circle there corresponds on the helicoid a point at a distance u from the z axis along a straight line generator whose projection on the xy plane makes an angle (v=c) with the x axis. The following table obtained by means of this relation shows the correspondence of the curves on the two surfaces.

On the Catenoid.		On the Helicoid.	
s = o $\bar{v}$ a variable	minimal circle.	u = o v a variable	z axis.
s = constant $\bar{v}$ a variable	a parallel circle.	u = constant v a variable	a helix
s a variable $\bar{v}$ = constant	a catenary.	u a variable v = constant	a straight line generator.

Therefore the minimal parallel circle on the catenoid corresponds to the z axis of the helicoid while in general parallel circles correspond to helices and catenaries to straight line generators. These same results may be obtained in

(1) Stegemann Article 88 - Aufgabe 7.

one step as follows: the catenoid expressed in u. v parameters in given by the equations

$$x = \sqrt{u^2+k^2}\cos v, \quad y = \sqrt{u^2+k^2}\sin v, \qquad z = k\log\left(u \pm \sqrt{u^2+k^2}\right)$$

When v is a constant the curves of these equations are catenaries and when u is a constant the curves are circles. A summary of these results shows that <u>in general catenaries and parallel circles on the catenoid correspond to straight lines and helices on the helicoid respectively</u> and in particular the minimal circle of the catenoid correspond to the axis of the helicoid.

As a result of these considerations, it appears that if the catenoid is cut thru the minimal parallel circle and one of the resulting halves cut along a meridian section, then bent out until the minimal circle becomes the straight line AB of the figure (5) and finally placed on the helicoid so that AB coincides with the z axis, the catenoid will fit exactly on the helicoid. The meridian sections of the catenoid will coincide with the generators of the helicoid and parallel circles will coincide with helices. Or from another point of view, a half of the catenoid may be thought of as being twisted until the minimal circle becomes a straight line in which case as in the figures (6) the resulting surface is a portion of a helicoid.

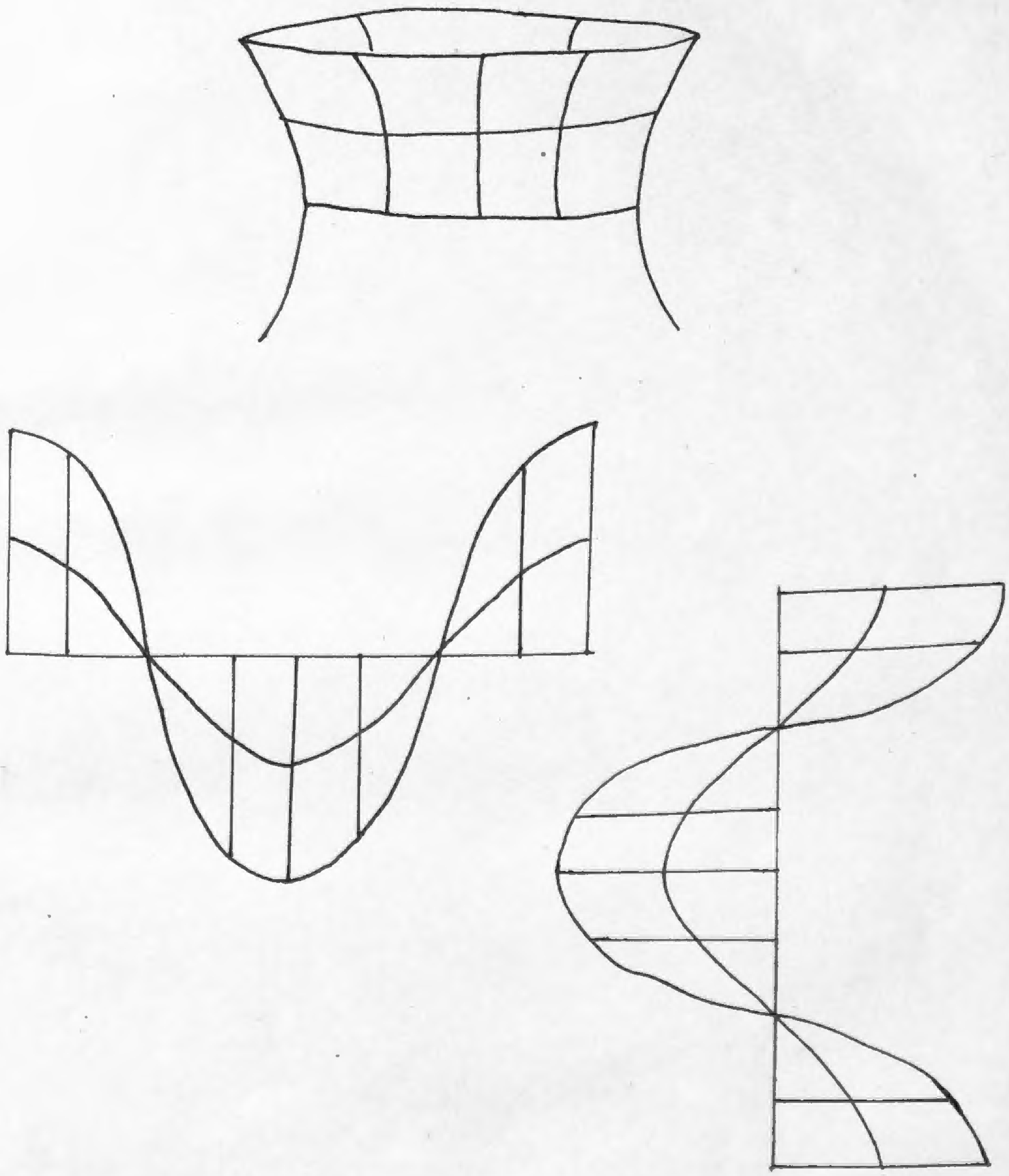


Figure (5)

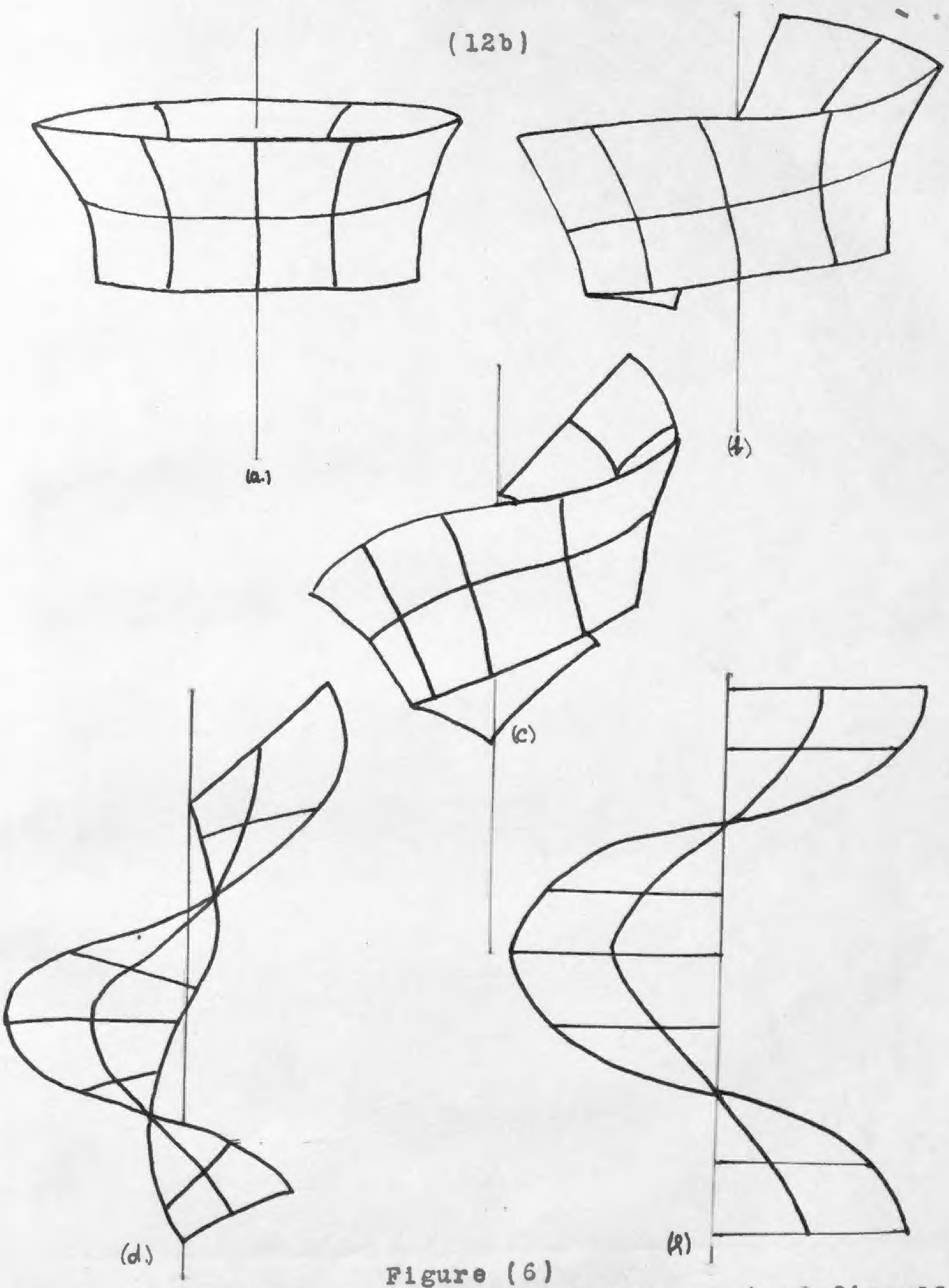


Figure (6)

Note: After this paper was completed a set of figures which suggested the figures (5) and (6) was found in the 1913 edition of Scheffers on page (363) Anflage II.

X. The Geodesics on the Catenoid.

The godesics on the catenoid are obtained by means of the formula

$$(14) \qquad x\frac{dy}{dt} - y\frac{dx}{dt} = c\frac{ds}{dt}.$$

which is the first integral for the problem of the geodesics on any surface of revolution whose equations are (11).[1] This first integral allows an important geometrical interpretation which is useful in the discussion of the geodesics and may be derived as follows. If in the formula (14) values from the equations (11) are substituted for x, $\frac{dy}{dt}$, y and $\frac{dx}{dt}$, the relation[1]

is obtained.

$$(15) \qquad \bar{u}^2\frac{d\bar{v}}{dt} = c\frac{ds}{dt}.$$

Now consider the figure (7) which may be the picture of any surface of revolution. Let denote the angle that an element of arc PQ of a geodesic makes with any parallel circle MPN. Then, if APB and A' Q B' are neighboring meridians, in the infinitessimal triangle PP'Q the cosine of has the value

$$(16) \qquad \cos\alpha = \frac{\bar{u}\,d\bar{v}}{ds}.$$

As a result of this equation (16), the equation (15) may be written in the form

$$(17) \qquad \bar{u}\cos\alpha = c.$$

[1]Kommerell und Kommerell, I Band page 116.& 117
[2]Joachimsthal, Article 111.

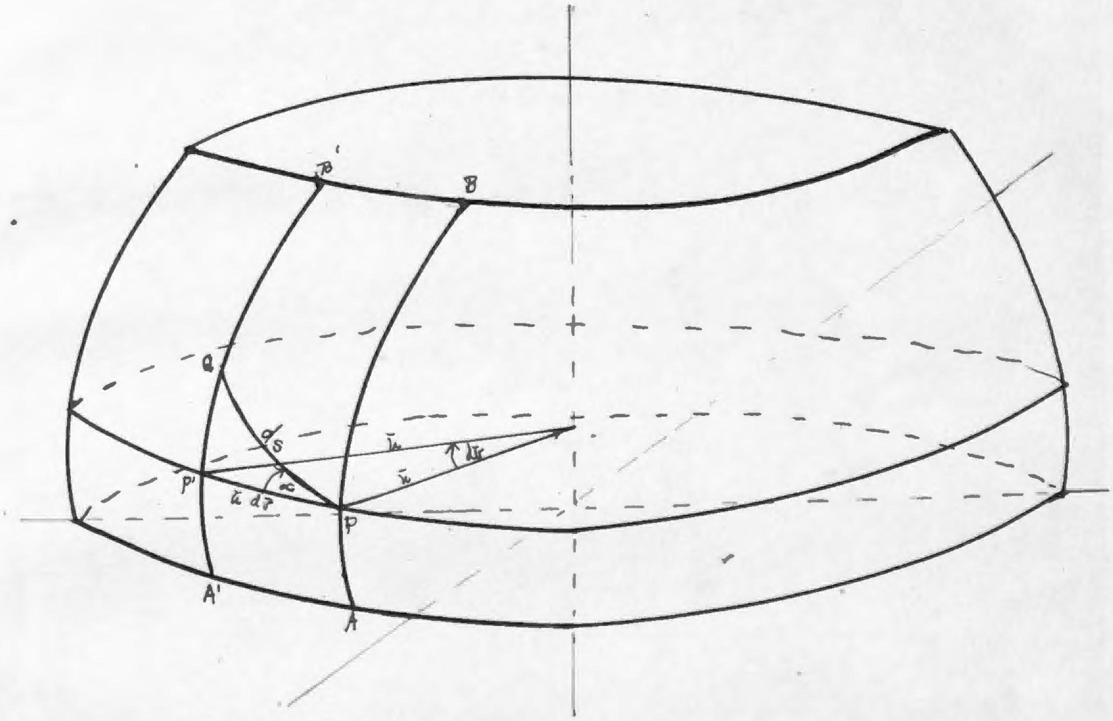


Figure (7)

Thus, it is evident that the value of $\bar{u}$ cannot be less than c i. e. the $\bar{u}$ coordinate of a point ona geodesic can never be less than the constant of integration. The equation (17) can also be given the following geometrical interpretation: at any point of a geodesic the product of the radius of a parallel circle and the cosine of the angle between the parallel circle and the geodesic through that point is constant.

One system of geodesics on the catenoid may be obtained at once directly from the equation (17). Since for the catenoid $\bar{u}$ equals or is greater than b, and since c is a constant of integration, a permissable set of values for the equation (17) is

$$\bar{u} \neq 0 \quad , \quad c = 0$$

which demands that cosine α equals zero. It appears from the equation (16) analytically and from the figure geometrically that the meridian sections i. e. the catenaries form a system of geodesics.

Other geodesics on the catenoid will now be obtained from the elliptic integral

$$(18) \qquad d\bar{v} = c \frac{d\bar{u}}{\sqrt{(\bar{u}^2 - c^2)(\bar{u}^2 - b^2)}}$$

which is the result of the substitution of the coordinates of the catenoid as given by equation (12) in thefirst integral (14). If the equation (18) is written in the form

$$c\, d\bar{u} = \sqrt{(\bar{u}^2 - c^2)(\bar{u}^2 - b^2)}\; d\bar{v}$$

one solution is seen at once to be the equation

$$\bar{u} = c$$

which represents the parallel circles. But since the only parallel circle which has the property characteristic of every geodesic, namely that its principal normals and the normals to the surface coincide[1] is the least circle, a geodesic is obtained only when c has the value b. Hence the minimal parallel circle on the catenoid is a geodesic.

The remaining study of the geodesics divides itself into three cases according as to whether in the elliptic integral (18) the constant of integration c is greater than less than or equal to b.

<u>Case A. $c > b$.</u>

If $c > b$ the substitution

$$\bar{u}^2 = c^2 \operatorname{cosec}^2 \varphi \tag{19}$$

reduces the equation (18) to an elliptic integral of the first class

$$\bar{v} = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = F(k, \varphi) \tag{20}$$

in which k has the value

$$k^2 = \frac{b^2}{c^2}$$

In order to visualize the curves of the equations (19) and (20) upon the catenoid, it will be of advantage to consider the following:

a) As seen by the equation (19) $\bar{u}$ has a period of π.

[1] Eisenhart, Article 57.

b). If the complete integral

$$\bar{v} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$$

is denoted by V, then an increase of n $\frac{\pi}{2}$ φ for φ increases $\bar{v}$ by n V. In particular to the period of π for $\bar{u}$, there corresponds a period of 2V = 2F (k,$\frac{\pi}{2}$) for $\bar{v}$. As c approaches b, k approaches unity and therefore 2V approaches infinity. Also as c approaches ∞, k approaches zero and 2V approaches π.

c) The following table of values computed from the equations (19) and (20)

$$\varphi = \frac{\pi}{2}, \qquad \bar{v} = V, \qquad \bar{u} = c$$

$$\varphi = \frac{\pi}{2} + \varphi_1, \quad 0 < \varphi_1 < \frac{\pi}{2}, \qquad \bar{v} = V + F(k,\varphi_1), \qquad \bar{u} = c_1$$

$$\varphi = \frac{\pi}{2} - \varphi_1, \quad 0 < \varphi_1 < \frac{\pi}{2}, \qquad \bar{v} = V - F(k,\varphi_1), \qquad \bar{u} = c_1$$

shows that for points on a geodesic differing from $\bar{v}$ equal to $\bar{V}$ by F (k, φ_1) $\bar{u}$ has the same values, i. e. $\bar{u}$ is symmetrical with respect to $\bar{v}$ = V.

d) From the equations (17) and (19) the relation

$$\varphi + i = \frac{\pi}{2}$$

is seen to hold. When φ approaches $\frac{\pi}{2}$ from the equation (19) $\bar{u}$ approaches c and from the equation (17) approaches zero. As φ approaches zero, $\bar{u}$ approaches infinity, while i approaches $\frac{\pi}{2}$. Geometrically this means that a geodesic is tangent to the parallel circle

$$\bar{u} = c$$

and that as $\bar{u}$ approaches infiniaty the geodesic approaches perpendicularity to the parallel circles or in other words approaches catenaries asymtotically.

e) From these considerations it is clear that the picture of one of these geodesics on the catenoid is that of the figures (8) and (9).

f) The projection of the geodesic of the figure (8) on the xy plane is seen to be the curve of the figure (10).

g) Since in general 2V radians is incommensurable with 2π radians and since c is any constant greater than b, a geodesic similar to that of the figure (8) branches out from every point on any parallel circle except the least.

h) For a point on the catenoid ($\bar{u}$, $\bar{v}$) the equations (17) gives for $\mathcal{C}$ the value

$$\mathcal{C} = \cos^{-1}\frac{c}{\bar{u}}$$

As the constant of integration c approaches its minimum and maximum values b and $\bar{u}$, $\mathcal{C}$ approaches as its limits $\cos^{-1}\frac{b}{\bar{u}}$ and zero degrees. Since c may assume an infinite number of values between b and $\bar{u}$, through any point not on the least circle there passes a pencil of geodesics.

<u>Case B. c<b.</u>

In this case the substitution

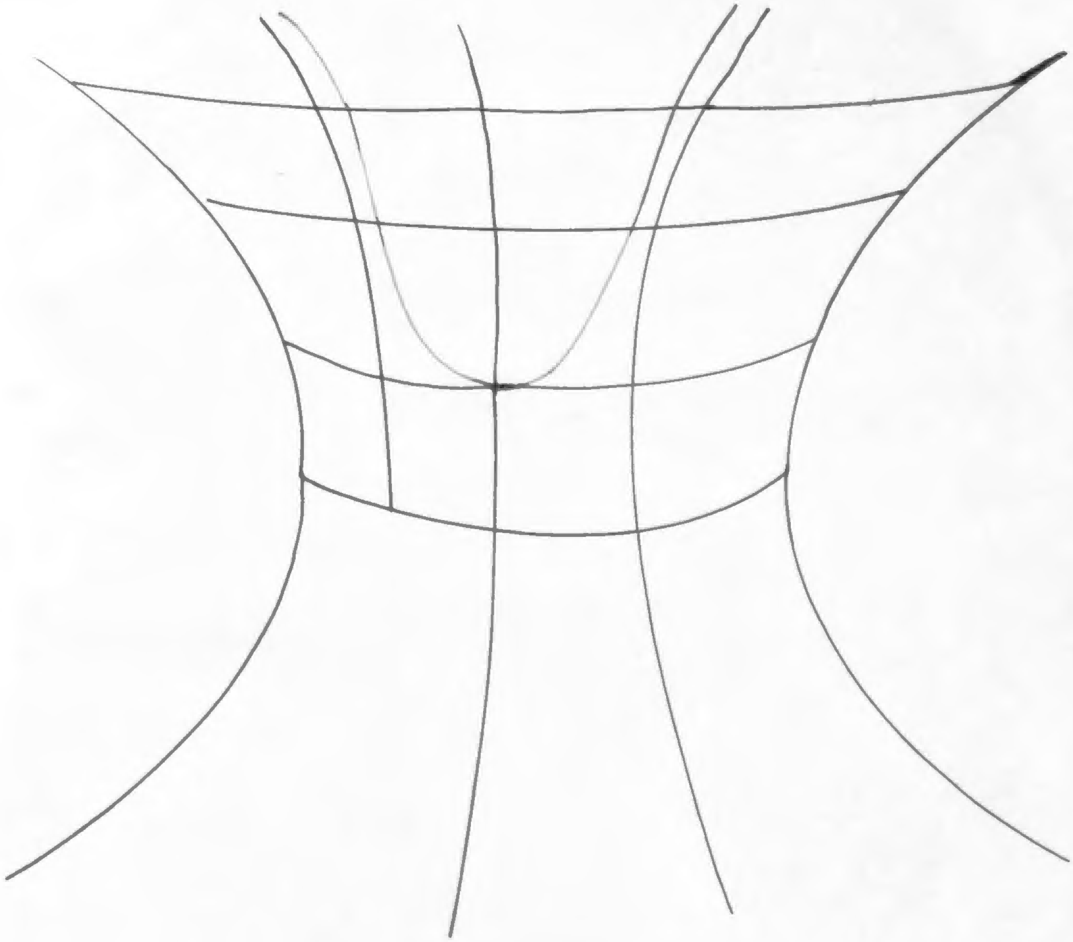


Figure (8)

Note: This would be the picture of the geodesic if the period 2V were less than π .

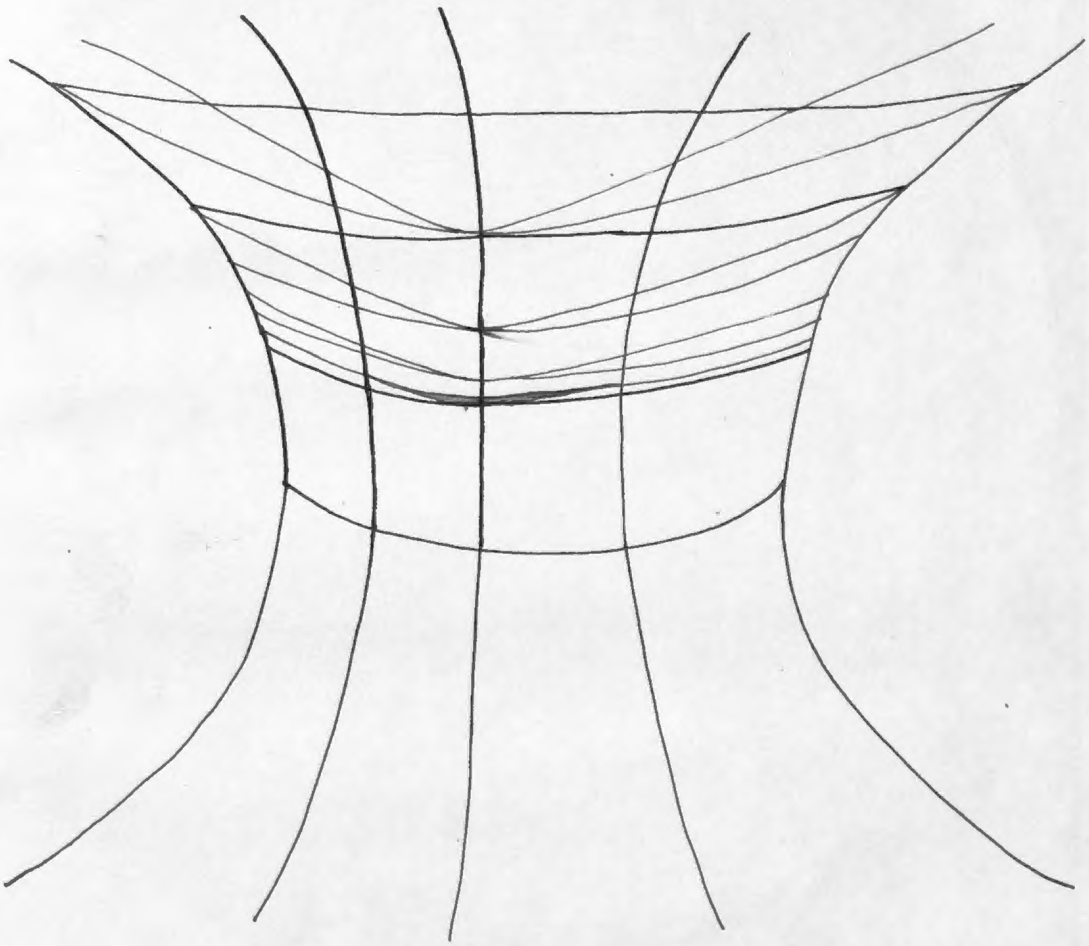


Figure (9).

Note: This is the picture of the geodesic with the size of the period taken into consideration.

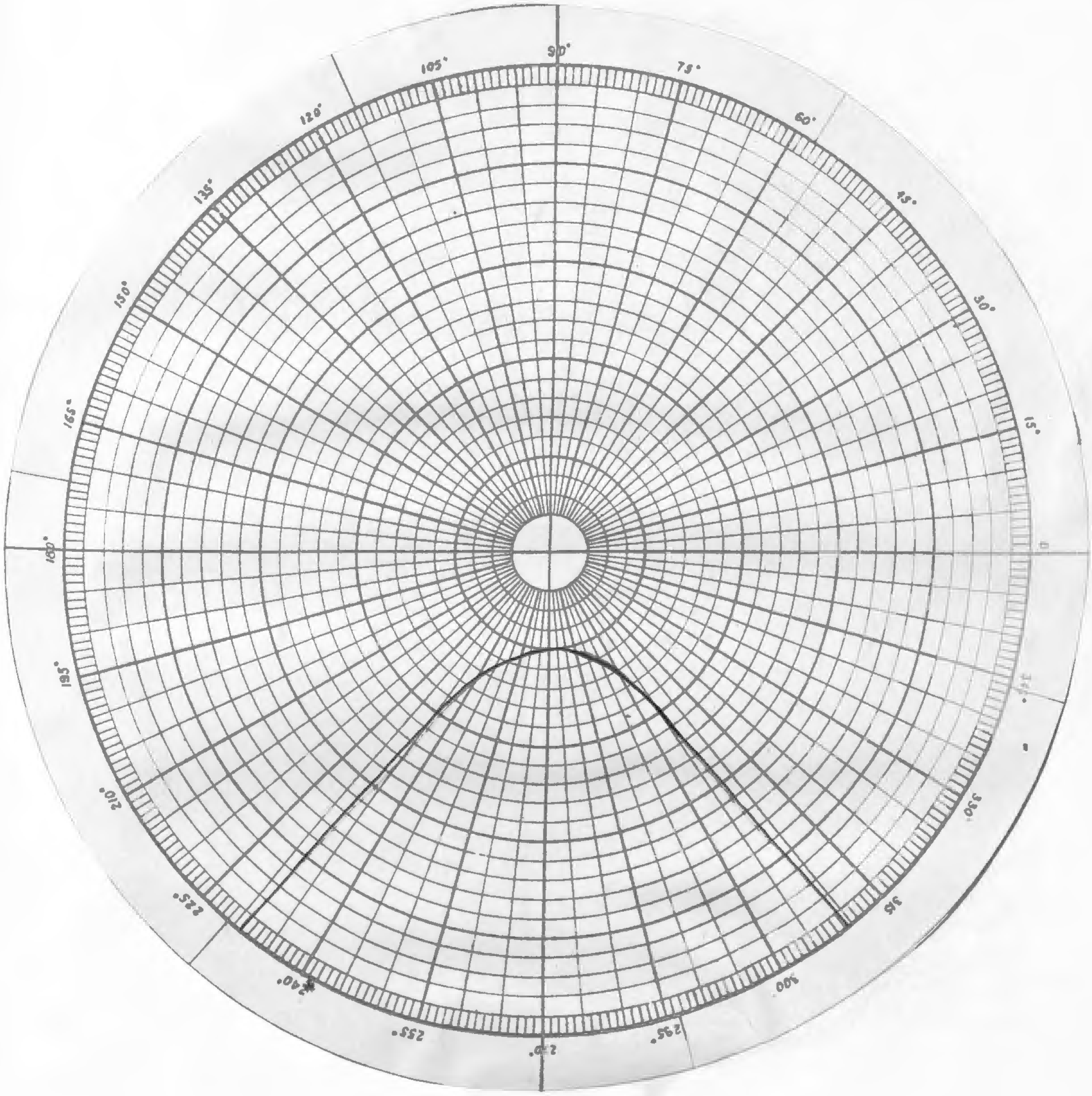

90°
105°
75°
120°
60°
135°
45°
150°
30°
165°
15°
180°
0
195°
345°
210°
330°
225°
315
240°
300
255°
285°
270°

Figure 10.

(21) $$\bar{u}^2 = b^2 \csc^2 \varphi$$

reduces the equation (18) to the elliptic integral

(22) $$\bar{v} = -k_1 \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

in which k has the value

$$k = \frac{c}{b}$$

The visualization of geodesics represented by these equations (21) and (22) will be made by a method similar to that of case A.

a) As in case A, $\bar{u}$ has a period of π.

b) Similar to case A, $\bar{v}$ has a period of $-2kV$ corresponding to the period of π for $\bar{u}$. Also as c approaches b, k approaches unity and 2kV approaches infinity. As c approaches zero, k approaches zero and 2kV approaches zero i. e. the geodesic itself approaches a catenary.

c) Similar to case A, $\bar{u}$ is symmetrical with respect to $\bar{v}$ equal kV.

d) From the equation (17) it is evident that for a point on any $\parallel \odot$, $\bar{u}$ = constant, α is $\cos^{-1} \frac{c}{\bar{u}}$. Thus through the point in question there pass two geodesics one lying in the first and one in the fourth quadrant and more over these geodesics cut the parallel circle. As $\bar{u}$ approaches infinity, α approaches $\frac{\pi}{2}$ and again the geodesic approaches catenaries as asymtotes, These meridian sections form an angle of $-2kV$ with one another.

e) As a result of these facts it is clear that the picture of a geodesic on the catenoid is that of the

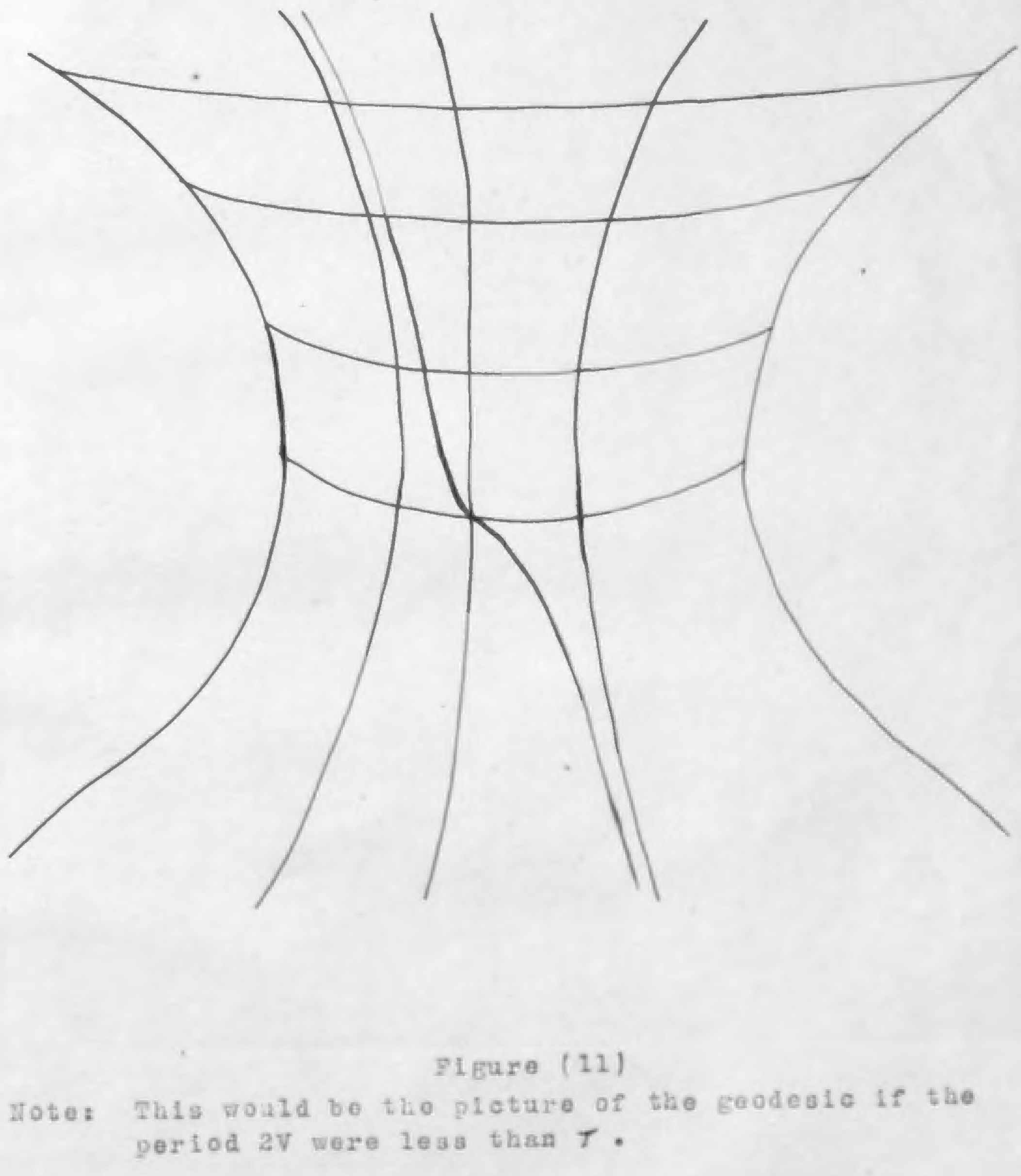


Figure (11)

Note: This would be the picture of the geodesic if the period 2V were less than π.

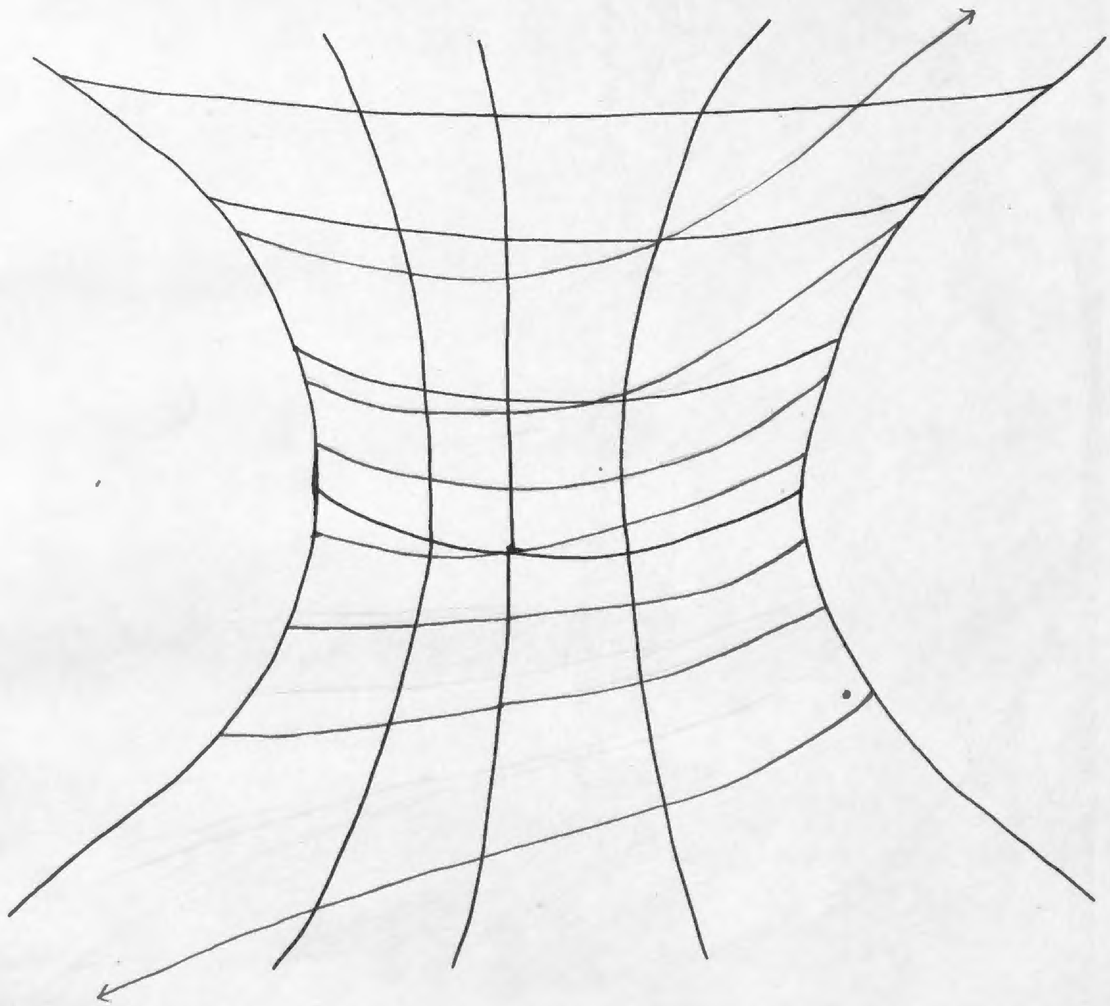


Figure (12)

Note: This is the picture of the geodesic with the size of the period 2V taken into consideration.

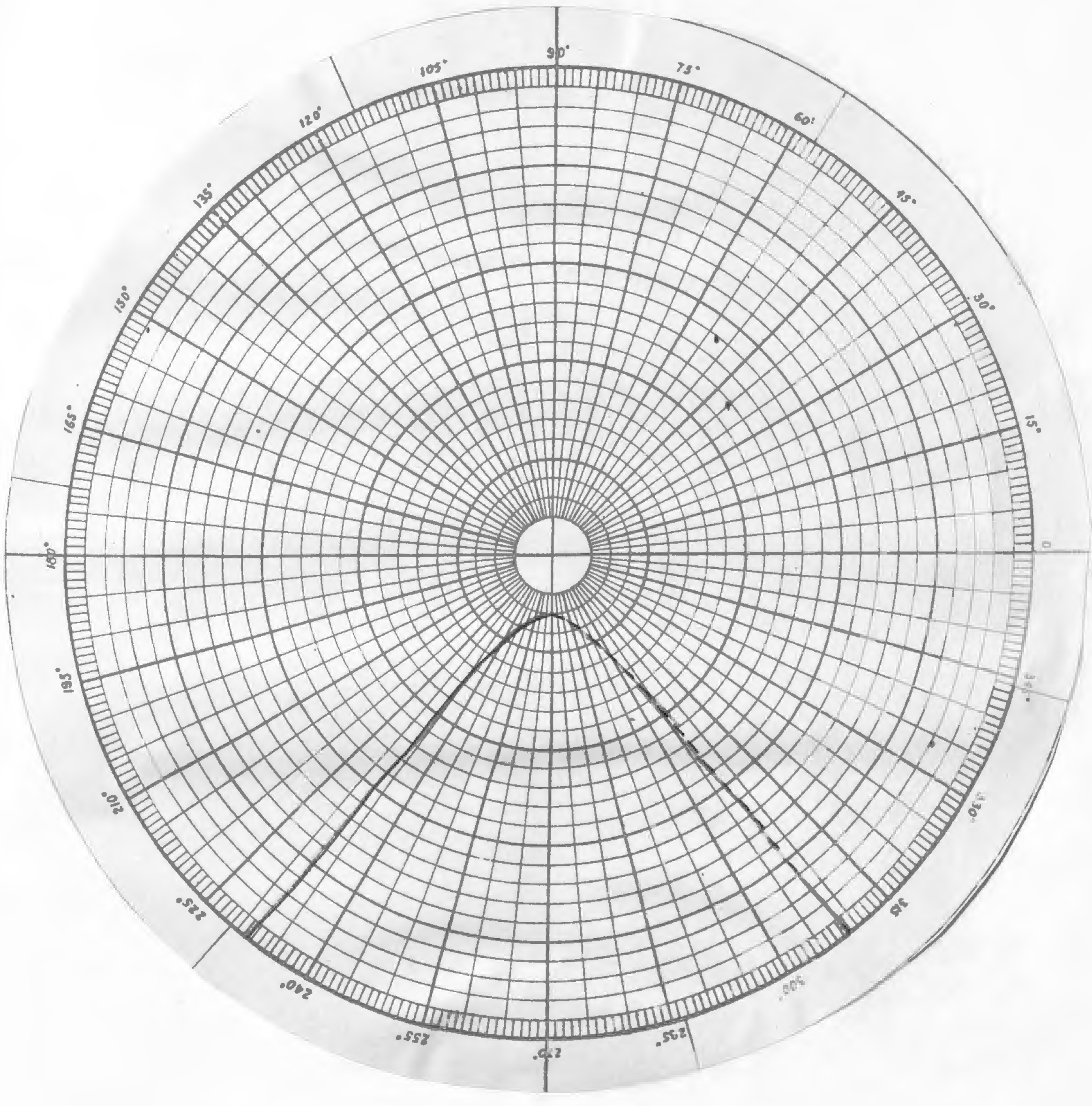

90°
105°
75°
120°
60°
135°
45°
150°
30°
165°
15°
180°
0
195°
210°
330°
225°
315
240°
300°
255°
285°
270°

Figure 13.

figures (11) & (12)

f). The projection of the geodesic on the xy plane is the graph of the figure (12) in which the dotted line is the projection of that part of the geodesic which is on the lower half of the catenoid.

g) Since in general 2kV radians are incommensurable with 2π radians and since c is less than b, a geodesic similar to that of the figure (11) crosses the minimal parallel circle at every point. ~~The figure (13) shows the projections of some of the infinite number of geodesics.~~

h) For a point on the catenoid $(\bar{u}, \bar{v})$ the equation (17) gives for α the value

$$\alpha = \cos^{-1} \frac{c}{\bar{u}}$$

As the constant of integration c approaches its minimum and maximum values zero and b, α approaches as its limits $\frac{\pi}{2}$ and $\cos^{-1} \frac{b}{\bar{u}}$. Since c may assume an infinite number of values between zero and b through a point on any parallel circle there passes a pencil of geodesics.

Case C. c=b.

When c equals b, the integral of the equation (18) has the form

$$d\bar{v} = c \frac{d\bar{u}}{\bar{u}^2 - b^2}. \tag{23}$$

A substitution

$$\sin\theta = \frac{b}{\bar{u}}$$

may be used in which case the equation (23) may be written as

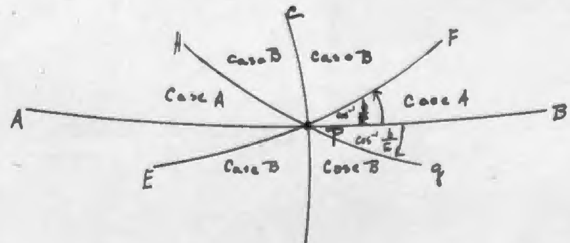
$$\bar{v} = \int_0^{\varphi} \frac{d\varphi}{\cos\varphi} = \log tg \left(\frac{\pi}{4} + \frac{\varphi}{2} \right)$$

or

(24) $$\tanh \bar{v} = \sin \varphi = \frac{b}{\bar{u}}$$

This last equation (24) may easily be graphed on the catenoid. When $\bar{v}$ approaches zero, $\bar{u}$ is infinite and as $\bar{v}$ approaches infinity $\bar{u}$ approaches b. Hence as in figure (14) the geodesic is a spiral which starting at an infinite point on a catenary winds down around the catenoid and approaches the minimal circle. The projection of the curve of the figure (14) in the xy plane is at once seen to be the spiral of the figure (15). Since in this case q equals b the value of i given by equation (17) is $\cos^{-1}\frac{b}{u}$, a fixed quantity for a definite point on any parallel circle. Thus through a point on any parallel circle two geodesics pass under $\cos^{-1}\frac{b}{u}$, one in the first and one in the fourth quadrant.

It is interesting to note that since the angle which the geodesics of this case make with the parallel circles is $\cos^{-1}\frac{b}{u}$, they form the boundary between the regions in which the geodesics of cases A and B lie. The accompanying diagram shows the regions in which the geodesics of cases A and B may lie. APB is a segment of a parallel circle CPD, a segment of a meridian section and EPF and GPH are the geodesics of case three through the point P.



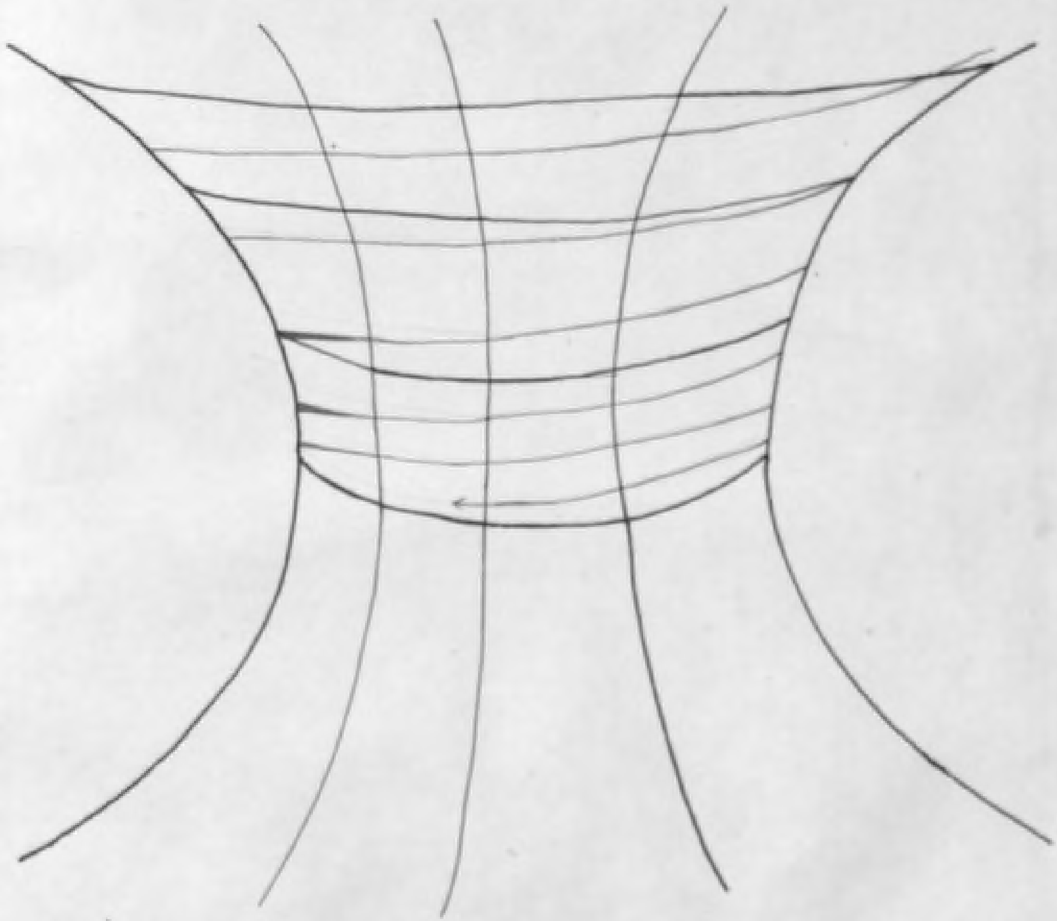


Figure (14)

(20-b)

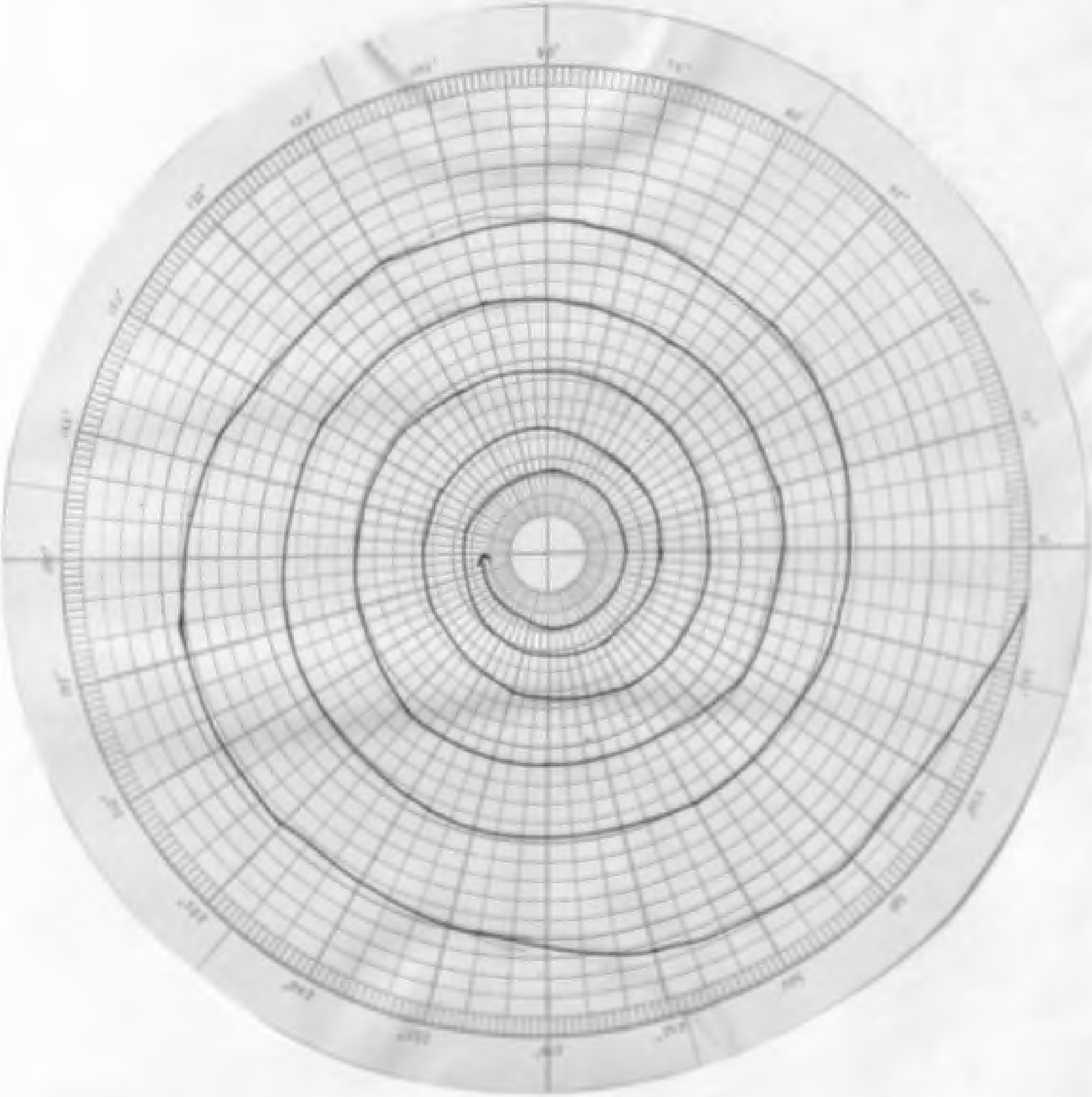


Figure (15).

XI. The Geodesics on the Helicoid.

It is now evident as a result of the applicability of the catenoid and the helicoid studied in part IX that the geodesics on the helicoid are the straight line generators, the z axis and one of the three following systems of curves. The geodesic of case A on the catenoid becomes on the helicoid a curve which is tangent to a helix and branching off from either side of the point of tangency approaches as asymtotes generating lines as in the figure (16). There are an infinite number of such curves one of which is tangent to every point on any helix. The geodesic of case B becomes on the helicoid a curve which starts off from the axis under an angle and approaches a generating line asymtotically as in the figure (17). There are an infinite number of such curves starting out from every point on the z axis. Finally the geodesic of Case C becomes on the helicoid a spiral which starts at an infinite point on a generator and twisting down along the helicoid the z axis as in the figure (18). Through any point on the helicoid two such curves pass and these form the boundaries between the geodesics of cases A and B which pass through the point in question.

Figure (16)

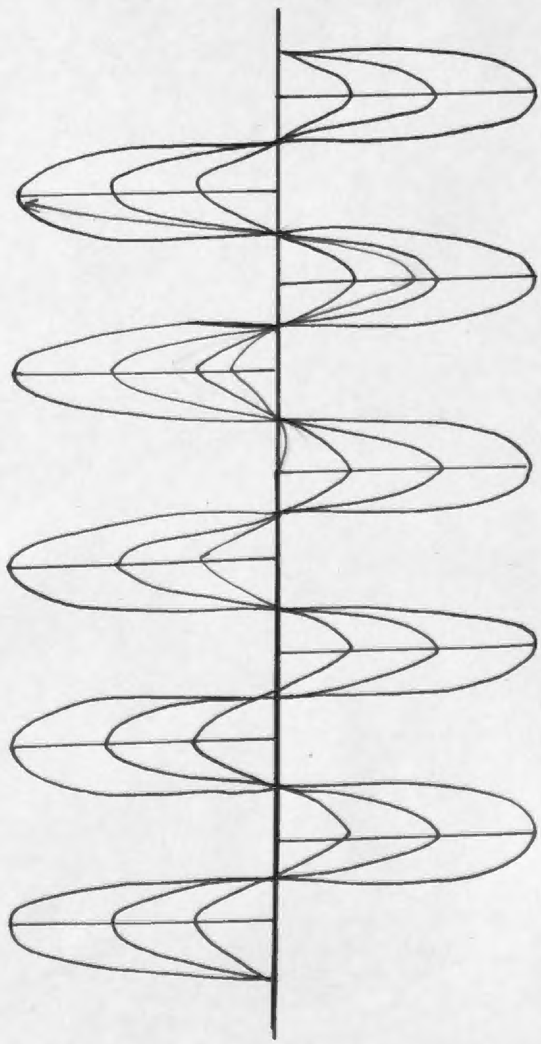


Figure (17) m

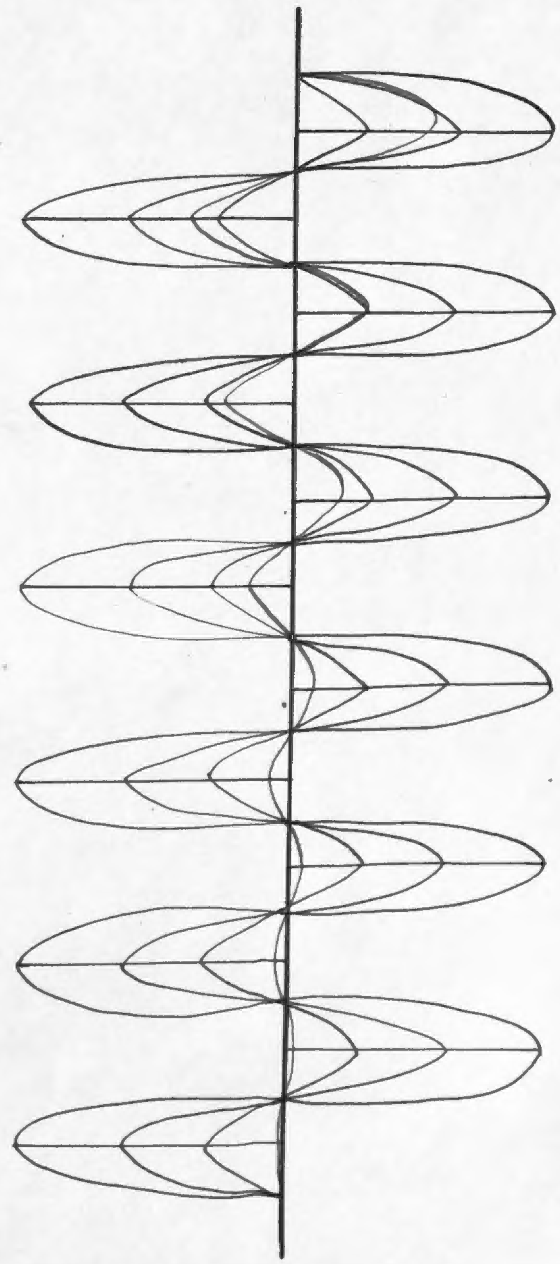


Figure (10)